JOURNAL OF APPROXIMATION THEORY 31, 22–32 (1981)

On the Existence of Hermite–Birkhoff Quadrature Formulas of Gaussian Type*

NIRA DYN

Department of Mathematical Sciences, Tel-Aviv University, Ramat-Aviv, Tel-Aviv, Israel

Communicated by G. G. Lorentz

Received August 1, 1979

Existence of quadrature formulas of Gaussian type related to Hermite–Birkhoff interpolation is proved for a class of incidence matrices satisfying the conditions of the Atkinson–Sharma Theorem. For the subclass of Hermite matrices this analysis furnishes yet another proof of the existence of Gaussian quadrature formulas with multiple nodes.

1. INTRODUCTION AND PRELIMINARIES

This paper studies the existence of quadrature formulas of Gaussian type related to Hermite–Birkhoff interpolation problems. Given the $m \times n$ incidence matrix $E = (e_{ij})_{i=1}^{m}, {}_{j=0}^{n-1}$ with entries consisting of zeros and ones, having precisely N ones, $N < n$, we are interested in the existence of a quadrature formula of the form:

$$\int_a^b f\, d\sigma \simeq \sum_{e_{ik}=1} a_{ik} f^{(k)}(x_i), \qquad a \leqslant x_1 < x_2 < \cdots < x_m \leqslant b, \tag{1.1}$$

which is exact for Π_{n-1}, the space of all polynomials of degree $\leqslant n-1$. Incidence matrices with $N < n$ ones which admit such quadrature formulas are termed in [3] "matrices of Gaussian type." It is proved in [3] that the quadrature formula (1.1) can be exact for Π_{n-1} only if $n \leqslant N + k$, where k is the minimal number of ones which must be added to E to obtain a matrix without odd sequences in rows corresponding to interior points of $[a, b]$. Two classes of matrices of Gaussian type admitting quadrature formulas (1.1) exact for Π_{n-1}, with $n = N + k$, are known:

* Accomplished on sabbatical at the Mathematics Research Cneter, University of Wisconsin, Maddison. Sponsored by the United States Army under Contract DAAG29-75-C-0024.

0021-9045/81/010022-11$02.00/0

(a) The class of Hermite matrices (matrices consisting of sequences of ones starting at column 0—Hermite sequences) with all the sequences corresponding to points in (a, b) of odd order [2, 6].

(b) The class of incidence matrices derived from quasi-Hermite matrices with Hermite sequences of length 2 in rows $2, \ldots, m-1$, by changing the last one in each of these sequences into zero [11]. (The definition of quasi-Hermite matrices is given at the end of this section.)

In the following we characterize a wide class of incidence matrices of Gaussian type for which $n = N + k$, where k equals the number of Hermite sequences corresponding to points in (a, b). This class contains the classes (a), (b). Our analysis is based on the Atkinson–Sharma theorem [1] and on the existence of the classical Guassian quadrature formulas (termed also principal representations) for Chebychev systems [8]. The idea of proof is in some respect an extension of the approach of Markov [10] to the construction of the Gaussian quadrature formulas (GQF) in the classical sense. In this approach the GQF is derived by integration of the Hermite interpolation polynomial:

$$p_{2k-1}(t; X^*) = \sum_{i=1}^{k} [f(x_i^*)\, p_{i0}(t; X^*) + f'(x_i^*)\, p_{i1}(t; X^*)],$$

$$p_{ij} \in \Pi_{2k-1}, \qquad p_{ij}^{(s)}(x_r) = \delta_{ir}\delta_{js},$$

$$s = 0, 1, r = 1, \ldots, k, j = 0, 1, i = 1, \ldots, k,$$

with $X^* = \{a \leqslant x_1^* < \cdots < x_k^* \leqslant b\}$ chosen so that

$$\int_a^b p_{i1}(t; X^*)\, dt = 0, \qquad i = 1, \ldots, k.$$

Although our method of proof does not yield uniqueness of the quadrature (1.1) even for matrices of class (a), for which uniqueness is known [2, 6], the proof of existence is somewhat simpler than the proofs in [2, 6]. Moreover by the same method it is possible to extend the uniqueness result to the case of quadrature formulas related to quasi-Hermite matrices.

The results and proofs are stated for polynomials, but the extension to extended complete Chebyshev systems is straightforward. (See Section 3.)

In Section 2 we prove the existence of quadrature formulas of Guassian type related to a certain class of incidence matrices. Section 3 consists of remarks on some extensions and on certain interesting specific cases.

We conclude this section by introducing notations and citing some results from the theory of Hermite–Birkhoff interpolation. Let $AS_{m \times n}$ denote the class of incidence matrices $E = (e_{ij})_{i=1}^{m}, {}_{j=0}^{n-1}$ satisfying:

(a) Pólya conditions

$$\sum_{j=0}^{n-1} \sum_{i=1}^{m} e_{ij} = n, \qquad \sum_{j=1}^{s-1} \sum_{i=1}^{m} e_{ij} \geqslant s, \qquad s = 1,..., n-1. \tag{1.2}$$

(b) All the non-Hermite sequences in rows $2,..., m-1$ are even. (A sequence is a maximal string of ones in a row.)

The well-known Atkinson–Sharma theorem [1] states that the interpolation problem at (E, X):

$$p^{(j)}(x_i) = \alpha_{ij}, \qquad e_{ij} = 1, \qquad p \in \Pi_{n-1}, \tag{1.3}$$

has a unique solution for any $X = \{a \leqslant x_1 < x_2 < \cdots < x_m \leqslant b\}$ and any data $\{\alpha_{ij}, e_{ij} = 1\}$ (E is order poised) if E satisfies (1.2) and if E contains no sequence $e_{i,k+1} = e_{i,k+2} = \cdots = e_{i,k+r} = 1$ with r odd, such that $e_{\nu\mu} = 1$ for some $\nu < i$, $\mu \leqslant k$ and for some $\nu > i$, $\mu \leqslant k$.

The Atkinson–Sharma theorem implies that all AS matrices are order poised. A subclass of particular interest of the AS matrices is the quasi-Hermite matrices satisfying (1.2) and containing only Hermite sequences in rows $2,..., m-1$. The solution to the interpolation problem (1.3) with $\alpha_{ij} = f^{(j)}(x_i)$ for $e_{ij} = 1$, and $f \in C^{n-1}[a, b]$ depends continuously on $X = \{a \leqslant x_1 < \cdots < x_m \leqslant b\}$. This solution can be extended continuously to all $X = \{a \leqslant x_1 \leqslant \cdots \leqslant x_m \leqslant b\}$ [13] if for X with one or more equal nodes (1.3) is replaced by

$$p^{(j)}(\hat{x}_i) = f^{(j)}(\hat{x}_i), \qquad \hat{e}_{ij} = 1, \qquad p \in \Pi_{n-1}. \tag{1.4}$$

In (1.4) $\hat{X} = \{a \leqslant \hat{x}_1 < \cdots < \hat{x}_l \leqslant b\}$ consists of all the distincts nodes of X, and $\hat{E}$ is obtained from E by coalescing rows of E corresponding to equal nodes of X according to the rule [5, 4]: For $\hat{x}_s = x_i = \cdots = x_{i+k}, \hat{e}_{sj} = 1$ if and only if

$$\sum_{\mu=r}^{j} \sum_{\nu=i}^{i+k} e_{\nu\mu} \geqslant j - r + 1 \qquad \text{for some} \quad 0 \leqslant r \leqslant j. \tag{1.5}$$

A quadrature formula of the form (1.1), exact for Π_{n-1}, is termed hereafter a "Hermite–Birkhoff Gaussian quadrature formula" (HB–GQF) if

$$N = \sum_{j=0}^{n-1} \sum_{i=1}^{m} e_{ij} = n - \sum_{\{i \mid x_i \in (a,b)\}} e_{i0}. \tag{1.6}$$

This is the best possible in case all rows of E corresponding to points in (a, b) contain Hermite sequences. In this case the number of parameters (points and coefficients) in (1.1) equals the dimension of the space for which (1.1) is exact.

2. Existence of Hermite–Birkhoff Gaussian Quadrature Formulas

A key observation in the forthcoming analysis is the following direct result of the Atkinson–Sharma theorem (see also [9, p. 89]):

LEMMA 1. *Let E be an $m \times n$ incidence matrix satisfying*:

$$e_{i0} = 0, \qquad 1 \leqslant i \leqslant m, \tag{2.1}$$

$$\sum_{j=0}^{s-1} \sum_{i=1}^{m} e_{ij} \geqslant s - r, \qquad s = 1, 2, \dots, n-1, \qquad \sum_{j=0}^{n-1} \sum_{i=1}^{m} e_{ij} = n - r, \tag{2.2}$$

for some positive integer r. If all sequences of E are even then for any $X = \{a \leqslant x_1 < x_2 < \cdots < x_m \leqslant b\}$ the subspace

$$P_0(E, X) = \{p \mid p \in \Pi_{n-1}, p^{(j)}(x_i) = 0, e_{ij} = 1\} \tag{2.3}$$

is a Chebychev space of dimension r on $[a, b]$.

It can be further shown that

LEMMA 2. *Under the assumptions of Lemma* 1, *$P_0(E, X)$ depends continuously on X in the open simplex*

$$S^m = \{X \mid X = a < x_1 < \cdots < x_m < b\}\}.$$

Proof. Let $p_1, \dots, p_r$ be the unique polynomials in Π_{n-2} interpolating $g(x) \equiv x^{n-1}$ according to the following r interpolation problems:
For fixed $a < y_1 < \cdots < y_r < b$

$$\left.\begin{array}{ll} P_k^{(j)}(x_i) = g^{(j)}(x_i), & e_{ij} = 1 \\ P_k(y_i) = g(y_i), & i = 1, \dots, k-1, k+1, \dots, r \end{array}\right\} k = 1, \dots, r. \tag{2.4}$$

Now $p_1, \dots, p_r$ depend continuously on $X \in S^m$, and for $i = 1, \dots, r$ $x^{n-1} - p_i \in P_0(E, X)$. Moreover the r functions $x^{n-1} - p_i$ $i = 1, \dots, r$ are linearly independent by (2.4). Therefore $P_0(E, X) = \operatorname{span}\{x^{n-1} - p_1, \dots, x^{n-1} - p_r\}$, and $P_0(E, X)$ depends on $X \in S^m$ continuously.

Remark 1. The definition of $P_0(E, X)$ can be extended continuously to the closure of S^m, $\overline{S^m} = \{X \mid X = \{a \leqslant x_1 \leqslant \cdots \leqslant x_m \leqslant b\}\}$, if for X on the boundary of $\overline{S^m}$

$$P_0(E, X) \equiv P_0(\hat{E}, \hat{X}),$$

where $\hat{X}$ consists of the distinct nodes of X, and $\hat{E}$ is obtained from E by coalescing rows of E corresponding to equal components of X, as in (1.5). This is the case since $\hat{E}$ satisfies the conditions of Lemma 1 whenever E does, and hence $P_0(E, X) \equiv P_0(\hat{E}, \hat{X})$ is a Chebychev space of dimension r on $[a, b]$. Morevoer by the continuity of the interpolation problems (2.4) in the case of coalescing rows [13], $p_1, ..., p_r$ and therefore $P_0(E, X)$ depend continuously on X in $\overline{S^m}$.

Similarly we have

LEMMA 3. *Let E be an $m \times n$ incidence matrix satisfying* (2.1) *and* (2.2). *If all sequences of E in rows $2, ..., m-1$ are even, then for any $X = \{a = x_1 \leqslant x_2 \leqslant \cdots \leqslant x_{m-1} \leqslant x_m = b\}$ the subspace $P_0(E, X)$ is a Chebychev space of dimension r, which depends continuously on X.*

Analogous results hold if all rows of E except the first (the last) consist of even sequences only. In this case $x_1 = a$ $(x_m = b)$ and the rest of the nodes vary in $[a, b]$ appropriately.

The main result of this work is the content of the next theorem,

THEOREM 1. *Let $E = (e_{ij})_{i=1}^{m}, {}_{j=0}^{n-1}$ be an incidence matrix with k Hermite sequences $0 < k \leqslant m$, satisfying*

$$\sum_{j=0}^{s-1} \sum_{i=1}^{m} e_{ij} \geqslant s - k, \qquad s = 1, 2, ..., n-1, \qquad \sum_{j=0}^{n-1} \sum_{i=1}^{m} e_{ij} = n - k. \tag{2.5}$$

If all the Hermite sequences of E are odd and all the non-Hermite sequences of E are even, then E admits an HB-GQF with nodes $X^* = \{a < x_1^* < \cdots < x_m^* < b\}$ for any positive measure supported on more than k points in (a, b).

Proof. Let

$$I_E \equiv \{i_1 < i_2 < \cdots < i_k\} \equiv \{i \mid e_{i0} = 1, 1 \leqslant i \leqslant m\}, \tag{2.6}$$

and let $\tilde{E}$ be obtained from E by replacing the first one in rows $i_1, ..., i_k$ by zero. It follows from (2.5) that $\tilde{E}$ satisfies the conditions of Lemma 1 with $r = 2k$ and therefore by Lemmas 1, 2, and Remark 1, $P_0(\tilde{E}, X)$ is a Chebychev space of dimension $2k$ for any $X \in \overline{S^m}$, with a continuous dependence on X in $\overline{S^m}$.

Using the result about the existence of a unique Gaussian quadrature formula (lower principal representation) for a Chebychev space [8, p. 47, Corollary 3.1] we conclude the existence of unique $Z = \{a < z_1 < \cdots < z_k < b\}$ and $w_i > 0$, $i = 1, ..., k$, such that

$$\int_a^b p \, d\sigma = \sum_{i=1}^{k} w_i \, p(z_i), \qquad p \in P_0(\tilde{E}, X), \tag{2.7}$$

for any positive measure $d\sigma$ supported on more than k points in (a, b). Moreover w_i, z_i, $i = 1,..., n$ are continuous functions of the Chebychev space for any fixed positive measure $d\sigma$.

Hence the mapping T_1 from $\overline{S^m}$ into S^k, which maps $X \in \overline{S^m}$ into the point $Z \in S^k$ of the nodes of the quadrature formula (2.7) for $P_0(\tilde{E}, X)$, is continuous.

In the following we construct a continuous mapping of the symplex $\overline{S^k}$ into itself. Let T_2 be any continuous mapping from S^k into S^m such that for $Y = \{a < y_1 < \cdots < y_k < b\}$, $X = T_2 Y$ is of the form:

$$x_{i_j} = y_j, \qquad j = 1,..., k, \qquad a < x_1 < x_2 < \cdots < x_m < b. \tag{2.8}$$

and let us extend this mapping continuously to $\overline{S^k}$. In case $k = m$ (all rows of E contain Hermite sequences) T_2 is the identity mapping. Thus the mapping $T \equiv T_1 T_2$ is a continuous mapping of $\overline{S^k}$ into itself, and therefore by the Brouwer fixed-point theorem [12] there exists $Y^* \in \overline{S^k}$ such that

$$TY^* = Y^*.$$

Moreover, since for all $Y \in \overline{S^k}$, $TY = Z \in S^k$, $Y^* = \{a < y_1^* < \cdots < y_k^* < b\}$, and therefore $X^* = T_2 Y^*$ is of the form $X^* = \{a < x_1^* < \cdots < x_m^* < b\}$. We conclude the proof of the theorem by showing that E admits an HB-GQF at the nodes X^*.

For $1 \leqslant i \leqslant m$ define μ_i as the length (number of ones) of the Hermite sequence in row i of E ($\mu_i = 0$ if $i \notin I_E$). Then the matrix $\hat{E} = (\hat{e}_{ij})_{i=1}^{m}{}_{,}{}_{j=0}^{n-1}$, obtained from E by replacing the k zeroes $e_{i\mu_i}$, $i \in I_E$, by ones, satisfies (1.2) in view of (2.5), and all its sequences are even. Hence $\hat{E} \in AS_{m \times n}$ and for any $\hat{e}_{ij} = 1$ there exists a unique polynomial p_{ij} satisfying

$$\frac{d^r}{dx^r} p_{ij}(x_s^*) = \delta_{is} \delta_{jr}, \qquad e_{rs} = 1, \qquad p_{ij} \in \Pi_{n-1}. \tag{2.9}$$

In particular $p_{l\mu_l} \in P_0(\tilde{E}, X^*)$ for $l \in I_E$.

Now since $TY^* = Y^*$, the points $x_{i_1}^*,..., x_{i_k}^*$ are the nodes of the GQF (2.7) for $P_0(\tilde{E}, X^*)$ and therefore

$$\int_a^b p_{l\mu_l} \, d\sigma = \sum_{s=1}^{k} w_s^* p_{l\mu_l}(x_{i_s}^*) = 0, \qquad l \in I_E. \tag{2.10}$$

Representing $p \in \Pi_{n-1}$ as the sum

$$p(t) = \sum_{\hat{e}_{ij}=1} p^{(j)}(x_i^*) \, p_{ij}(t) \tag{2.11}$$

and integrating, we obtain in view of (2.10) the quadrature formula

$$\int_a^b p\,d\sigma = \sum_{\substack{\hat{e}_{ij}=1 \\ j\neq\mu_i}} a_{ij}\,p^{(j)}(x_i^*) = \sum_{e_{ij}=1} a_{ij}\,p^{(j)}(x^*), \qquad p\in \Pi_{n-1}, \tag{2.12}$$

with

$$a_{ij} = \int_a^b p_{ij}\,d\sigma, \qquad e_{ij}=1.$$

Remark 2. In the classical approach to GQF, for given odd integers $\mu_1,..., \mu_m$, $\sum_{i=1}^m (\mu_i+1)=n$, a set of nodes X^* is found such that the coefficients $a_{i\mu_i}$, $i=1,..., m$, in the quadrature formula

$$\int_a^b p\,d\sigma = \sum_{i=1}^{m}\sum_{j=0}^{\mu_i} a_{ij}\,p^{(j)}(x_i^*), \qquad p\in \Pi_{n-1}, \tag{2.13}$$

all vanish. Theorem 1 implies that for any given set of indices $\{j_1,..., j_m\}$ with $1\leqslant j_i\leqslant \mu_i$, j_i odd, $i=1,..., m$, there exists a quadrature formula of the form (2.13) with $a_{ij_i}=0$, $i=1,..., m$.

More generally a matrix E admits an HB-GQF if it is obtained from an order-poised matrix $\hat{E}$ consisting of even sequences only, by replacing a one in an odd column in each of the Hermite sequences of $\hat{E}$ by zero.

As in the case of GQF with multiple nodes [2, 6], also in the HB-GQF (2.12) certain coefficients are positive. Using the notation of Theorem 1 we prove:

COROLLARY 1. *In the HB-GQF* (2.12), $a_{ij}>0$ *for* $i\in I_E$, j *even* $j<\mu_i$.

Proof. For $i\in I_E$, j even $j<\mu_i$, let q_{ij} be the polynomial solving the following interpolation problem:

$$q_{ij}^{(s)}(x_\nu^*)=0, \qquad \hat{e}_{\nu s}=1, \quad \nu\neq i, \tag{2.14}$$

$$q_{ij}^{(s)}(x_i^*)=0, \qquad \hat{e}_{is}=1, \quad s\neq j,\mu_i, \tag{2.15}$$

$$q_{ij}^{(j)}(x_i^*)=1, \tag{2.16}$$

$$q_{ij}(a)=0. \tag{2.17}$$

Then by (2.12) $\int_a^b q_{ij}\,d\sigma = \sum_{e_{\nu s}=1} a_{\nu s} q_{ij}^{(s)}(x_\nu^*) = a_{ij}$.

To prove that $a_{ij}>0$ it is sufficient to show that $q_{ij}\geqslant 0$, $q_{ij}\not\equiv 0$ on the support of $d\sigma$. We first show that $q_{ij}(x)\neq 0$, $x\in(a,b)-\{x_\nu^* \mid \nu\in I_E\}$, and that the zeros of q_{ij} in $\{x_\nu^* \mid \nu \in I_E\}$ are even. Suppose to the contrary that either $q_{ij}(\xi) = 0$ for $\xi \in (a, b) - \{x_\nu^* \mid \nu \in I_E\}$ or that $q_{ij}^{(\mu_\nu+1)}(x_\nu^*) = 0$ for some $\nu \in I_E$, $\nu \neq i$. Then q_{ij} is a nontrivial solution of a homogeneous interpolation problem, corresponding to a matrix $\bar{E} = (\bar{e}_{ij})_{i=1}^{m},{}_{j=0}^{n-1}$ obtained from $\hat{E}$ by substituting $\bar{e}_{ij} = 0$, $\bar{e}_{i\mu_i} = 0$, adding a row with 1 in the first column corresponding to the point a, and either adding a row with 1 in its first column corresponding to the point ξ, or substituting $\bar{e}_{\nu,\mu_\nu+1} = 1$. In either cases $\bar{E} \in AS_{m\times n}$—a contradiction. Thus q_{ij} vanishes in (a, b), only at points of $\{x_\nu^* \mid \nu \in I_E\}$, and all these zeros even by (2.14)–(2.16) and the structure of $\hat{E}$. In view of (2.16) $q_{ij} \geqslant 0$, while $q_{ij} \not\equiv 0$ on the support of $d\sigma$, since $\{x_\nu^* \mid \nu \in I_E\}$ contains k points.

The same idea of proof can be extended to matrices satisfying conditions as in Theorem 1 but with no restrictions on the structure of the first and/or the last row. We state the result and sketch the proof for one such case:

THEOREM 2. *Let $E = (e_{ij})_{i=1}^{m},{}_{j=0}^{n-1}$ be an incidence matrix with k Hermite sequences $0 < k \leqslant m-2$ in rows $2,\dots, m-1$, satisfying* (2.5). *If all the Hermite sequences in rows $2,\dots, m-1$ are odd and all the non-Hermite sequences of E in these rows are even, then E admits an HB-GQF with nodes $X^* = \{a = x_1^* < x_2^* < \cdots < x_m^* = b\}$, for any positive measure $d\sigma$ supported on more than k points in (a, b).*

Proof. Let

$$I_E \equiv \{i_1 < \cdots < i_k\} \equiv \{i \mid e_{i0} = 1, 2 \leqslant i \leqslant m-1\}$$

and let $\tilde{E}$ be derived from E by replacing the first one in each Hermite sequence of E by zero (including Hermite sequences in rows $1, m$). Then $\tilde{E}$ satisfies the conditions of Lemma 3 with $r = 2k + e_{10} + e_{m0}$, and therefore $P_0(\tilde{E}, X)$, with $X = \{a = x_1 \leqslant x_2 \leqslant \cdots \leqslant x_m = b\}$, is a Chebychev space of dimension $2k + e_{10} + e_{m0}$, for all $(x_2,\dots, x_{m-1}) \in \bar{S}^{m-2}$.

We define a mapping T from $\bar{S}^k$ into S^k as follows: For $Y \in S^k$ let $X = \{a = x_1 < \cdots < x_m = b\}$, with $x_2,\dots, x_{m-1}$ defined as in (2.8), and for $Y \in \bar{S}^k$ define $X \in \bar{S}^m$ by extending continuously this mapping. Then $TY = Z \in S^k$, where $Z = \{a < z_1 < z_2 < \cdots z_k < b\}$ are the interior points of the GQF (principal representation) for $P_0(\tilde{E}, X)$, corresponding to the measure $d\sigma$, which involves a if $e_{10} = 1$ and b if $e_{m0} = 1$ [8, p. 46]. Using the fixed point of T, $Y^* \in S^k$, we construct $x_2^* < \cdots < x_{m-1}^*$ by (2.8), and proceed as in the proof of Theorem 1 to construct the HB-GQF corresponding to E at the nodes $X^* = \{a = x_1^* < x_2^* < \cdots < x_m^* = b\}$. As in the case of Theorem 1 we have:

COROLLARY 2. *Let E be defined as in Theorem 2, and let*

$$\int_a^b p\,d\sigma = \sum_{e_{ij}=1} a_{ij}\, p^{(j)}(x_i^*), \quad p \in \Pi_{n-1}$$

be the corresonding HB-GQF. Then $a_{ij} > 0$ for $i \in I_E$, j even, $j < \mu_i$, $a_{1j} > 0$ for $j < \mu_1$, and $(-1)^j a_{mj} > 0$ for $j < \mu_m$, where μ_i is the length (number of ones) of the Hermite sequence in row i of E.

3. EXTENSIONS AND REMARKS

3.1. Let E in Theorem 2 consist of Hermite sequences of length 1 in rows $2,..., m-1$. Then $\tilde{E}$ has nonzero elements only in the first and last rows, $\sum_{j=0}^{n-1} (e_{1j} + e_{mj}) = n - 2(m-2)$, and for all $X = \{a = x_1 \leqslant \cdots \leqslant x_m = b\}$

$$P_0(\tilde{E}, X) = P_0(\tilde{E}) = \{p \mid p \in \Pi_{n-1}, p^{(j)}(a) = 0, e_{1j} = 1, j > 0, \\ p^{(j)}(b) = 0, e_{mj} = 1, j > 0\} \tag{3.1}$$

is a Chebychev space of dimension $d = 2(m-2) + e_{10} + e_{m0}$. Since the space $P_0(\tilde{E}, X)$ is independent of $x_2,..., x_{m-1}$, the points $x_2^*,..., x_{m-1}^*$ of Theorem 2 are the interior points of the GQF for $P_0(\tilde{E})$ involving a if $e_{10} = 1$ and involving b if $e_{m0} = 1$. Thus in this special case the derivation of the HB-GQF for E does not invpolve the construction of a fixed point of a mapping. In addition the same arguments yield the uniqueness of the HB-GQF corresponding to E.

3.2. A direct consequence of the simple relation between the HB-GQF constructed in Subsection 3.1 for the matrix E, and a certain GQF corresponding to $P_0(\tilde{E})$, is the following extremal property of this HB-GQF:

Let E be as in Subsection 3.1. Among all polynomials from

$$Q_n = \{q \mid q \in \Pi_n, q \geqslant 0, q^{(j)}(a) = 0, e_{1j} = 1, q^{(j)}(b) = 0, e_{mj} = 1\} \tag{3.2}$$

with leading coefficient $(-1)^s$, $s = \sum_{j=0}^{n-1} e_{mj}$, the one which has double roots at $x_2^*,..., x_{m-1}^*$ of Subsection 3.1 minimizes $\int_a^b q(x)\,dx$.

To see this observe that any $q \in Q_n$ with leading coefficient $(-1)^s$, can be written as $q = p_n - p$, where $p_n = (-1)^s x^n + \cdots$ satisfies

$$\begin{aligned} p_n^{(j)}(a) &= 0, \quad e_{1j} = 1, \quad j > 0, \\ p_n^{(j)}(b) &= 0, \quad e_{mj} = 1, \quad j > 0, \end{aligned} \tag{3.3}$$

and where $p \in P_0(\tilde{E})$ satisfies $p \leqslant p_n$ on $[a, b]$, $p(a) = p_n(a)$ if $e_{10} = 1$, and

$p(b) = p_n(b)$ if $e_{m0} = 1$. Since $\{p_n\} \cup P_0(\tilde{E})$ is a Chebychev space of dimension $d+1$, the extremal property of the GQF used in Subsection 3.1 yields the required result. (For extremal properties of GQF (principal representations) consult [8]). The results of Subsections 3.1 and 3.2 are the content of Theorem 4 in [11]. The proof of these results as sketched here, seems to be simpler.

It should be noted that GQF with multiple node (HB-GQF admitted by Hermite matrices) have a similar extremal property [7].

3.3. Theorems 1, 2 can be extended by the same method of analysis to the case of extended complete Chebyshev systems $\{u_1,..., u_n\} \subset C^{n-1}[a, b]$ is an extended complete Chebyshev system if any nontrivial "polynomial" $\sum_{i=1}^{k} a_i u_i$, $k \leqslant n$, has at most $k-1$ zeros counting multiplicities. The Atkinson–Sharma theorem is valied also for extended complete Chebyshev systems [5], but with the operators $(d^k/dx^k)\, k = 1,..., n-1$, replaced by certain differential operators $D_1,..., D_{n-1}$ depending on the extended Chebyshev system. In case of Hermite matrix E, the resulting HB-GQF involves only evaluations of the function and its derivatives of order at most $n-1$. Otherwise the HB-GQF is of the form:

$$\sum_{\substack{e_{ij}=1 \\ j<\mu_i}} a_{ij} f^{(j)}(x_i^*) + \sum_{\substack{e_{ij}=1 \\ j>\mu_i}} a_{ij}(D_j f)(x_i^*), \tag{3.4}$$

were $\mu_1,..., \mu_m$ are defined as in Section 2.

3.4. The existence and uniqueness of the HB-GQF admitted by an Hermite matrix E, in case of extended Chebyshev systems, is proved in [2, 6]. Using the uniqueness of the GQF with multiple nodes, we can prove the uniqueness of the HB-GQF related to a quasi-Hermite matrix by a construction similar to that in Subsection 3.1.

Indeed, let E in Theorem 2 consist of odd Hermite sequences of length $\mu_2,..., \mu_{m-1}$ in rows $2,..., m-1$, respectively, and let

$$\hat{P}_0(E) = \{p \mid p \in \Pi_{n-1}, p^{(j)}(a) = 0, e_{1j} = 1, j > 0,\\ p^{(j)}(b) = 0, e_{mj} = 1, j > 0\}. \tag{3.5}$$

By Lemma 2, $\hat{P}_0(E)$ is a Chebychev space of dimension $n - \sum_{j=1}^{n-1} (e_{0j} + e_{mj})$. Now

$$\int_a^b p\, d\sigma = \sum_{j=0}^{\mu_i - 1} a_{ij}\, p^{(j)}(x_i^*) + \sum_{e_{1j}=1} a_{1j}\, p^{(j)}(a) + \sum_{e_{mj}=1} a_{mj}\, p^{(j)}(b),$$

for all $p \in \Pi_{n-1}$ if and only if $x_2^*,..., x_{m-1}^*$ are the interior nodes of the unique GQF with multiple nodes for $\hat{P}_0(E)$, which involves a if $e_{10} = 1$ and involves b if $e_{m0} = 1$.

3.5. It is conjectured that for E in Theorem 1 with $k=m$ (all rows contain Hermite sequences) the corresponding HB-GQF is unique, as in the case of Hermite matrices [2, 6]. Uniqueness cannot be expected in the more general case due to the arbitrariness in the construction of the mapping T_2 of (2.8).

ACKNOWLEDGMENT

The author wishes to thank Professor S. D. Riemenschneider for a valuable discussion on the present general setting of the results in this work.

REFERENCES

1. K. ATKINSON AND A. SHARMA, A partial charcterization of poised Hermite–Birkhoff interpolation problems, *SIAM J. Numer. Anal.* **6** (1969), 230–235.
2. D. L. BARROW, On multiple node Gaussian quadrature formula, *Math. Comp.* **32** (1978), 431–439.
3. G. G. LORENTZ AND S. D. RIEMENSCHNEIDER, Birkhoff quadrature matrices, *in* "Linear Spaces and Approximation" (P. L. Butzer and Sz. -Nagy, Eds.), pp. 359–374, Birkhäuser Verlag, Basel, 1978.
4. G. G. LORENTZ AND S. D. RIEMENSCHNEIDER, Recent progress in Birkhoff interpolation, *in* "Approximation Theorey and Functional Analysis" (J. B. Prolla, Ed.), pp. 187–236, North-Holland, Amsterdam, 1979.
5. S. KARLIN AND J. M. KARON, Poised and non-poised Hermite–Birkhoff interpolations, *Indiana Univ. Math. J.* **21** (1972), 1131–1170.
6. S. KARLIN AND A. PINKUS, Gaussian quadrature formulae with multiple nodes, *in* "Studies in Spline Functions and Approximation Theory," pp. 113–141, Academic Press, London/New York, 1976.
7. S. KARLIN AND A. PINKUS, An extremal property of multiple Gaussian nodes, *in* "Studies in Spline Functions and Approximation Theory," pp. 142–162, Academic Press, London/New York, 1976.
8. S. KARLIN AND W. J. STUDDEN, "Tchebycheff Systems with Applications in Analysis and Statistics," Interscience, New York, 1966.
9. E. KIMCHI AND N. RICHTER-DYN, Best uniform approximation with Hermite–Birkhoff interpolatory side conditions, *J. Approx. Theory* **15** (1975), 85–100.
10. A. A. MARKOV, "Differenzenrechnung," Teubner, Leipzig (1896).
11. C. A. MICCHELLI AND T. J. RIVLIN, Quadrature formulae and Hermite–Birkhoff interpolation, *Advances in Math.* **11** (1973), 93–112.
12. J. M. ORTEGA AND W. C. RHEINBOLDT, "Iterative Solution of Nonlinear Equations in Several Variables," Academic Press, London/New York, 1970.
13. N. DYN, G. G. LORENTZ, AND S. D. RIEMENSCHNEIDER, Continuity of the Birkhoff interpolation, to appear.